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Structure of Growing Networks with no Preferential Attachment

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Abstract— **Based on the formation of triad junctions, the proposed mechanism generates growing networks that exhibit extended power law behavior and strong neighborhood clustering. The asymptotic behavior of both properties is of interest in the study of networks in which (i) the formation of links cannot be described according to the principle of preferential attachment; (ii) the in-degree distribution fits a power law for nodes with a high degree and an exponential form otherwise; and (iii) the degree of clustering depends on both the number of links that newly added nodes establish and the probability of forming triads.**

I. INTRODUCTION

Networks are systems composed of multiple-coupled, but well-defined elements (nodes) that display collective behaviors at higher levels of analysis. Large networks arise by the gradual addition of elements which attach to an existing and often evolving network component. With our modern access to data, the application of network techniques offers a wide set of mathematical tools to visualize data at both levels, that of the data elements and the interaction between them. These tools allow us to characterize higher-level properties of the structure of a system and to identify different types of patterns in the relationships among elements.

The development of models that describe the evolution of networks has been driven by the need to analyze large amounts of relational data across a wide range of fields. Well-known examples include the study of relationships we see in scientific collaborations [1], court opinions [2], export goods [3], traffic [4], social ties [5], stocks [6], and patent citations [7]-[9]. Trying to address the question of how particular topologies arise as networks grow, a large body of work has been devoted to understand the emergence of two properties: the distribution of links per node (degree distribution) and the proportion of links grouped into local neighborhoods (clustering or transitivity) [10], [11].

In extended power law networks, the probability p_k that a node with a low degree of connectivity (below some threshold ε) connects to k other nodes fits an exponential form $e^{-\lambda k}$ for some positive constant λ. For nodes with a high degree, the probability p_k is proportional to the power law function $k^{-\alpha}$ for some positive constant α. Because the tail of the probability distribution of the degree of the nodes has no exponential bound, the patterns of interaction in power law networks differ in orders of magnitude, with a few nodes being highly connected. Mechanisms leading to power law networks have been overviewed in [12]. A particular class of mechanisms in which nodes with a high degree have a greater probability of acquiring new links (attributed to the principle of preferential attachment) has been proposed to explain the scaling behavior in empirical data [13], [14].

In clustered networks, the probability of finding transitive triplets is higher than the outcome expected through random chance. If a node connects to two other nodes, clustering captures the probability that these two nodes are connected, too. In a network with high clustering, nodes do not interact homogeneously with other nodes, but tend to influence each other locally (i.e., they form strong neighborhood clusters [15]). Common measures of clustering are based on (i) the total number of transitive triplets relative to the total number of possible triplets in the network, represented by a global clustering coefficient C [11]; or (ii) the fraction of triplets connecting the neighboring nodes of node i over the total number of possible triplets, represented by a local clustering coefficient C_i [16]. Real-world networks show clustering coefficients that are generally independent of the size of the network and scale with the degree of the nodes [17].

Though preferential attachment offers an explanation for the existence of networks with power law degree distributions, it does not, by itself, explain the formation of strong neighborhood clusters. Clustering coefficients tend to vanish with the continuous addition of new nodes to a network (based on both local and global preferential attachment mechanisms [18]). The development of alternative models that can explain strong neighborhood clustering as the natural outcome of the process of growth contributes towards establishing a framework that supports the analysis and broadens our understanding of the clustering behavior of power law networks.

Based on the principle of preferential attachment, the authors of [19], [20] introduce a baseline probability of establishing additional links by a process of triad formation. They generate undirected networks with tunable degree distribution and clustering properties. In [20] the authors deduce analytical results based on conditions underlying local attachment mechanisms. Unlike [19], [20] the work in [21] explains power law behavior in networks in which the formation of links does not necessarily depend on preferential attachment. The attachment of new nodes results according to a uniform random distribution followed by the formation of triad junctions. Like [21] the formation mechanism in this paper does not instantiate the principle of preferential attachment, focusing on conditions that generate extended, rather than single, power law distributions [22],

[23].

The contribution of the proposed mechanism is twofold. First, it explains scaling behavior in networks with an extended power law in the in-degree distribution of the nodes (extended power laws are in some contexts a better fit than single or double power laws, e.g., to describe the degree distribution of online social networks [24] and patent citation networks [8]). Our results characterize the relation between the scaling exponent and the probability of forming triads. Moreover, the transition from exponential to power law distributions depends on both the scaling exponent and the number of links that newly added nodes establish. Second, the proposed mechanism accounts for strong neighborhood clustering based on a random triad formation process with a positive stationary mean. Clustering properties remain constant as the size of the network grows.

The remaining sections are organized as follows: Section 2 introduces a network model that captures generic connectivity dynamics. Theorem 1 in Section 3 shows that the resulting in-degree distribution follows a power law above a certain threshold ε and an exponential distribution otherwise. We present analytical results for the values of α, λ, and ε. Theorem 2 characterizes the evolution of both clustering coefficients and presents analytical expressions for C and C_i. Simulations in Section 4 capture the effect of triads on the scaling exponent and the clustering coefficients. Finally, Section 5 draws some conclusions and future research directions.

II. A NETWORK FORMATION MODEL

Let $\mathcal{H}_t = \{1, \ldots, N_t\}$ be a finite set of interconnected nodes at time index t. The set of edges $\mathcal{A}_t = \{(i,j) : i, j \in \mathcal{H}_t\}$ represents the relationships between nodes, where (i,j) indicates that there exists a directed edge between nodes i and j. Let $\mathcal{G}_t = (\mathcal{H}_t, \mathcal{A}_t)$ represent the network. Let $q_i(t) = \{j \in \mathcal{H}_t : (j,i) \in \mathcal{A}_t\}$ represent all nodes that link to node i at time t (i.e., its incoming neighbors). For any node $i \in \mathcal{H}_t$, let $k_i(t) = |q_i(t)|$ represent its in-degree.

A. Node attachment

The network grows by the gradual addition of nodes. Every time index t a new node attaches to m different nodes, selected according to a uniformly random distribution over $\mathcal{H}_{t-1}$. Let $n \geq 0$ denote the amount of edges established from nodes in $\mathcal{H}_{t-1}$ to the newly added node, according to some mechanism that responds to the attachment of the node. If there is no such response underlying the attachment process then $n = 0$.

B. Triad formation

Conditions for the formation of triad junctions are similar to the ones introduced in [19]. When node $j \notin \mathcal{H}_{t-1}$ attaches to some node $j' \in \mathcal{H}_{t-1}$, it may also establish an additional link to one of the outgoing neighbors of node j', selected again according to a uniformly random distribution. If $j \in q_{j'}(t)$ and $j' \in q_i(t)$, node j links to node i with probability $x_i(t)$. The value of $x_i(t)$ is subject to a multivariate random variable X_t with a positive expected probability $p_t = E[X_t] = \int \cdots \int f(\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_s) d\sigma_1\ d\sigma_2 \cdots d\sigma_s$, where $\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_s$ are independent factors that influence the formation of triads. Note that if all the outgoing neighbors of node j' are a subset of the outgoing neighbors of node j then there is no possibility of establishing additional links through triad formation. Let $X = \{X_t\}$ be the associated random process with stationary mean $p > 0$. The process of triad formation repeats itself for each edge established during the attachment step (m times) before another node joins the network.

Assumption 1 *(on the initial network)*: To ensure the two-steps of node attachment and triad formation can be properly completed, we require that (a) the network $\mathcal{G}_0$ is weakly connected; and (b) the network $\mathcal{G}_0$ has at least m nodes, each with at least one outgoing neighbor.

Assumption $1(a)$ is satisfied if replacing all the directed edges with undirected ones produces a connected undirected graph. Assumption $1(b)$ means that $N_0 \geq m$ and for every node $i \in \mathcal{H}_0$ there exists a node i' such that $i \in q_{i'}(0)$. This last condition is required when $p = 1$.

III. ANALYSIS

It is of interest that the proposed mechanism guarantees stationary values for both the in-degree distribution and the clustering coefficient of the network. The proofs of the following theorems are presented in the Appendix.

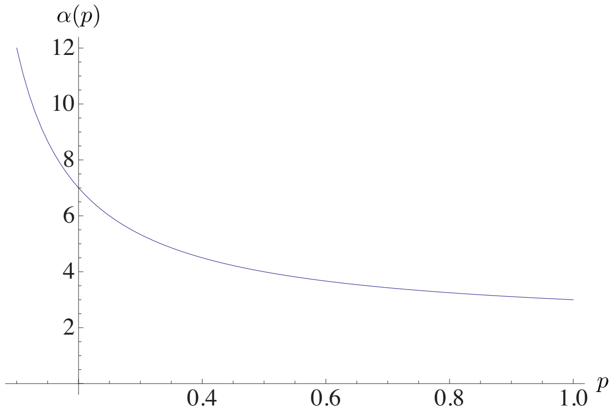
Theorem 1 *(in-degree distribution): For all $\mathcal{G}_0$ that satisfy Assumption* 1*, the in-degree distribution p_k of $\mathcal{G}_t$ follows an extended power law as $t \to \infty$. The scaling exponent $\alpha = 2 + \frac{1}{p}$ and the exponential exponent $\lambda = \frac{\alpha}{\alpha - 1}$ with threshold $\varepsilon = (\alpha - 1)\, m$.*

Theorem 1 implies that, as the network grows, the scaling exponent of the in-degree distribution depends on the stationary mean of the process of forming triads. The distribution follows a strict power law for nodes with a degree greater than $(\alpha - 1)\, m$ and an exponential fit otherwise. Figure 1(a) shows the value of the scaling exponent α for different values of p. Note that the the proposed mechanism allows $\alpha \geq 3$.

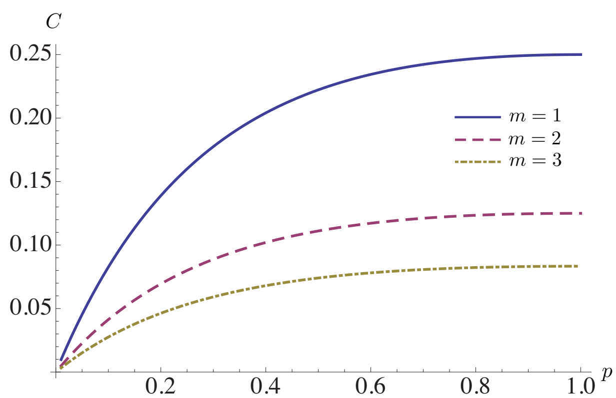
Theorem 2 *(clustering coefficients): For all $\mathcal{G}_0$ that satisfy Assumption* 1*, the global clustering coefficient of $\mathcal{G}_t$ tends to $C = \frac{p}{m(1+p)^2}$ as $t \to \infty$. The asymptotic behavior of the local clustering coefficient for a node with in-degree $k_i = k$ follows*

$$C_i(k) = \frac{2\left(k + pm + (2 + p - \alpha)\ln\left(\frac{k+\varepsilon}{n+\varepsilon}\right)\right)}{(k + p\varepsilon)\,(k + p\varepsilon - 1)}$$

Theorem 2 implies that C and C_i are both independent of the initial network configuration and the size of the growing network (i.e., clustering coefficients do not vanish). Figure 1(b) shows the value of the global clustering coefficient C for different values of m. Note that there exists an inverse relationship between the clustering behavior and the amount of edges established at every random attachment (m).

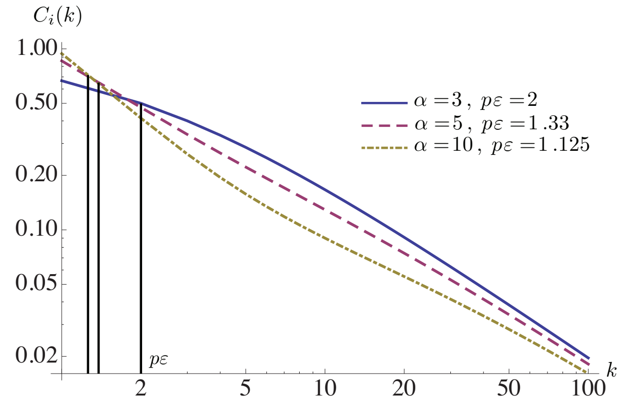


(a) Scaling exponent.

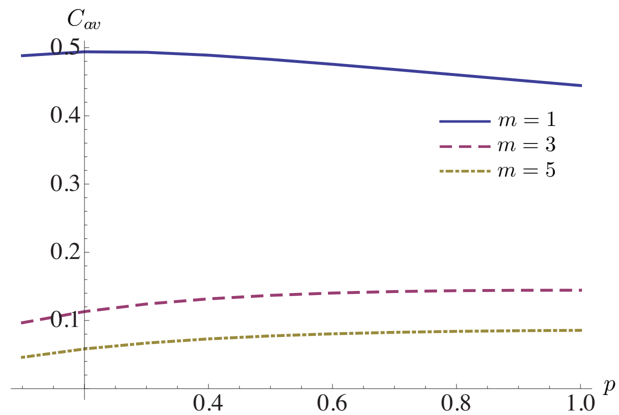


(b) Global clustering coefficient.

Fig. 1: (a) Scaling exponent α for different values of p; and (b) global clustering coefficient C for different values of p and m.



(a) Local clustering coefficient.



(b) Average clustering coefficient.

Fig. 2: (a) Local clustering coefficient $C_i(k)$ for different values of the scaling exponent α with $m = 1$ and $n = 1$; and (b) average clustering coefficient for different values of p and m with $n = 1$.

Moreover, note that for nodes with a low degree, high values of α tend to form strong neighborhood clusters (below $p\varepsilon$). Figure 2(a) shows the effect of α on the local clustering coefficient. For nodes with a high degree, the effect is opposite and the local clustering coefficient is proportional to k^{-1} (a behavior observed in empirical data [17]). Like for the global clustering coefficient, fig. 2(b) shows an inverse relationship between the average clustering coefficient $C_{av} = \int_n^\infty p_k C_i(k) dk$ and m. Finally, note that the average clustering coefficient is slightly greater than the global clustering coefficient (also observed in empirical measures of clustering [11]).

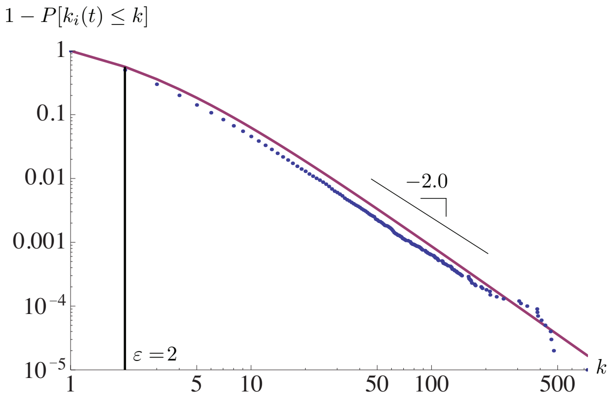
IV. SIMULATIONS

To gain insight into the outcome of the network formation process, let $N_0 = 12$, $n = 1$, $c = 0.1u$, and consider $t = 10^5$. Following similar ideas as in [21], [25], let the probability of establishing additional links due to triad formation be $x_i(t) = 1 - \frac{c}{uk_i(t)}$, where u captures the compatibility between nodes and is chosen from a uniformly random distribution with support on $[0, 1]$ (i.e., the random variable X_t takes values $x_i(t)$). The parameter c, $0 < c < u$, represents the cost of establishing additional links. The expected value of X_t at time t is given by

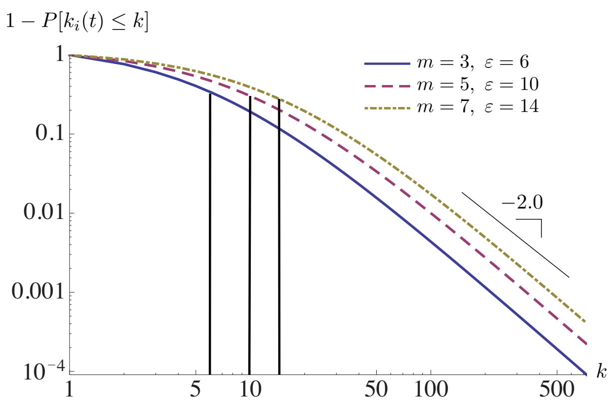
$$p_t = E[X_t] = \int_n^\infty \int_0^1 \left(1 - \frac{c}{uk_i(t)}\right) p_u p_k \, du \, dk_i \qquad (1)$$

where $p_u = \frac{1}{u}$ and p_k is the probability distribution of $k_i(t)$ presented in Theorem 1. It can be shown that (1) converges to 1 for $n > 0$ and $m > 0$. As a result, the process of triad formation has stationary mean $p = 1$.

Next, fig. 3 shows the in-degree distribution of the nodes of the network for different values of m. For nodes with a low degree, the complementary cumulative degree distribution deviates from the power law behavior with $\alpha = 3$ and degenerates into the exponential form with $\lambda = \frac{3}{2}$. In particular, $\varepsilon = 2m$ characterizes the transition from exponential to power law distributions.

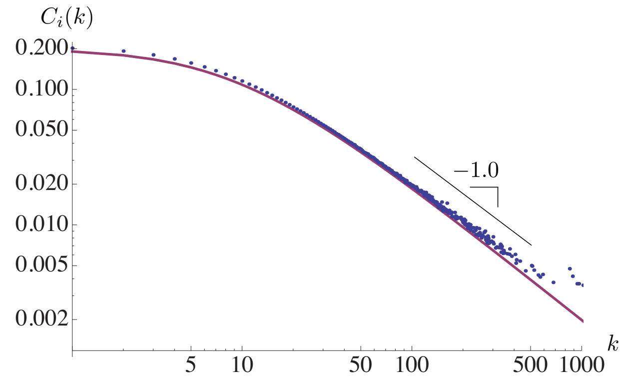


(a) $m = 1$.

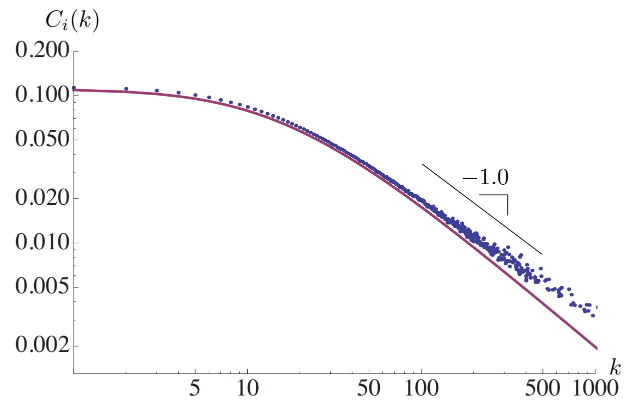


(b) $m = 3, m = 5, m = 7$.

Fig. 3: (a) Complementary cumulative distribution function of the in-degree distribution p_k on a logarithmic scale. The solid lines are the predictions from (4); the dots represent distributions from simulations; and (b) predictions from (4) for different values of m.



(a) $m = 3$.



(b) $m = 5$.

Fig. 4: Clustering coefficient $C(k)$ for (a) $m = 3$; and (b) $m = 5$. The solid lines are the predictions from (14); the dots represent the clustering coefficients of the generated network.

With respect to clustering, fig. 4 shows the value of the local coefficient $C_i(k)$ for different values of m. Note that for values greater than $2m$ the clustering behavior has a tendency $C_i(k) \sim k^{-1}$.

V. CONCLUSIONS

This paper introduces a mathematical framework that generates extended power law distributions with constant clustering coefficient based on two stages: (i) a node attachment step in which a newly added node links to a finite number of randomly selected nodes; and (ii) a triad formation step in which the new node may establish an additional link to one of the neighbors of the node it attaches to. The proposed mechanism is of interest because it helps explain the existence of extended power law networks with clustering properties that do not vanish as the size of the network grows. Network substrates with a desired scaling and clustering behavior allow us to evaluate which principles lie behind the formation of relationships in large amounts of data. The study of growth processes that lead to community structures on clustered networks provides an important direction for future research.

VI. APPENDIX

Proof of Theorem 1. We assume that the in-degree of node i is a continuous variable $k_i \in \mathbb{R},\ k_i > 0$. Every time step t a newly added node $j \notin \mathcal{H}_{t-1}$ attaches to m different nodes in $\mathcal{H}_{t-1}$, selected according to a uniform distribution process over the $N_0 + t - 1$ existing nodes. The probability that node j attaches at time t to an existing node $i \in \mathcal{H}_{t-1}$ with in-degree $k_i(t-1)$ is

$$\frac{m}{N_0 + t - 1}$$

The triad formation that follows the random attachment step adds to the rate of change of the in-degree of node i by

$$\left(\frac{mk_i(t)}{N_0 + t - 1}\right)\left(\frac{1}{m(1+p)}\right)p$$

The term $\frac{mk_i(t)}{N_0+t-1}$ is the probability of selecting, during the random attachment step, an incoming neighbor of node i (i.e., some node $j' \in q_i(t)$). Additionally, the term $\frac{1}{m(1+p)}$ is the probability that node j' is an incoming neighbor of node i (i.e., $j' \in q_i(t)$). Finally, the probability p is the stationary mean of the random process of forming triads. The multiplication of these three terms define the probability of forming a triplet with an edge contributing to the in-degree of node i. The overall rate of change of $k_i(t)$ is given by

$$\frac{dk_i(t)}{dt} = \frac{m}{N_0+t-1} + \frac{p}{1+p}\frac{k_i(t)}{N_0+t-1} \tag{2}$$

with boundary condition $k_i(t_i) = n$. The solution to (2) is

$$\begin{aligned} k_i(t) &= \left(n + \left(1+\frac{1}{p}\right)m\right)\left(\frac{N_0+t-1}{N_0+t_i-1}\right)^{\frac{p}{1+p}} \\ &\quad - \left(1+\frac{1}{p}\right)m \end{aligned} \tag{3}$$

Using (3), the analytical expression for the cumulative distribution of the in-degree $P[k_i(t) \leq k]$ of node i equals

$$\begin{aligned} & P\left[\left(n + \left(1+\frac{1}{p}\right)m\right)\left(\frac{N_0+t-1}{N_0+t_i-1}\right)^{\frac{p}{1+p}}\right. \\ & \left. - \left(1+\frac{1}{p}\right)m \leq k\right] \\ = \quad & P\left[t_i \geq \left(\frac{n+\left(1+\frac{1}{p}\right)m}{k+\left(1+\frac{1}{p}\right)m}\right)^{1+\frac{1}{p}} (N_0+t-1)\right. \\ & \left. - (N_0-1)\right] \end{aligned}$$

As $t \to \infty$

$$P[k_i(t) \leq k] = 1 - \left(\frac{n+\left(1+\frac{1}{p}\right)m}{k+\left(1+\frac{1}{p}\right)m}\right)^{1+\frac{1}{p}} \tag{4}$$

Finally,

$$p_k = \frac{dP[k_i(t) \leq k]}{dk} = a\left(k + \left(1+\frac{1}{p}\right)m\right)^{-\left(2+\frac{1}{p}\right)} \tag{5}$$

with $a = \left(1+\frac{1}{p}\right)\left(n+\left(1+\frac{1}{p}\right)m\right)^{1+\frac{1}{p}}$. Note that (5) exhibits an extended power law of the form

$$p_k \sim (k+\varepsilon)^{-\alpha}$$

with $\alpha = 2 + \frac{1}{p}$ and $\varepsilon = (\alpha-1)m$. When $k \gg \varepsilon$, (5) is reduced to a single power law $p_k \sim k^{-\alpha}$. On the other hand, when $k \ll \varepsilon$ we have

$$\begin{aligned} \ln p_k \sim -\alpha \ln(k+\varepsilon) &= -\alpha\left[\ln\left(1+\frac{k}{\varepsilon}\right) + \ln\varepsilon\right] \\ &\sim -\alpha\left[\frac{k}{\varepsilon} + \ln\varepsilon\right] \end{aligned}$$

and obtain

$$p_k \sim \varepsilon^{-\alpha}\exp\left(-\alpha\frac{k}{\varepsilon}\right)$$

Thus, (5) is proportional to the exponential form $p_k \sim \exp(-\lambda k)$ with $\lambda = \frac{\alpha}{\alpha-1}$.

□

Proof of Theorem 2. Note that the only configuration to form transitive triplets is when node $j \notin \mathcal{H}_t$ attaches to $j' \in \mathcal{H}_t$ such that $j \in q_{j'}(t)$ and there exists a node $i \in \mathcal{H}_t$ such that $j' \in q_i(t)$. A triad is formed if node j establishes a third edge to node i that connects nodes j, j', and i. The probability of establishing the third edge that closes the triplet is pm. Moreover, when node j entered the network, it connected to $m(1+p)$ outgoing neighbors (because node j established m edges according to the random attachment process and then established additional edges with probability pm according to the process of triad formation). Each outgoing neighbor of node j also has $m(1+p)$ outgoing neighbors. Thus, there are $m^2(1+p)^2$ possible pairs to form triplets. The global clustering coefficient is characterized by

$$C = \frac{pm}{m^2(1+p)^2} = \frac{p}{m(1+p)^2} \tag{6}$$

Next, to capture the local clustering coefficient of a node i, note that the number of possible pairs of incoming and outgoing edges of node i (with in-degree $k_i = k$) is given by

$$\begin{aligned} & \binom{k+m(1+p)}{2} \\ = \quad & \frac{(k+m(1+p))(k+m(1+p)-1))}{2} \end{aligned} \tag{7}$$

Equation (7) captures the total number of possible triplets that involve node i. Now, to capture the number of actual triplets that involve the node i, we consider three possible scenarios about the edges that may lead to triad formation: Node i has (i) two outgoing edges; (ii) an outgoing edge and an incoming edge established through random attachment; and (iii) two incoming edges with at least one of them having been generated through triad formation.

In scenario (i), there are an expected

$$pm \tag{8}$$

connected triplets.

In scenario (ii), the number of incoming edges created through random attachment is

$$\frac{dk_i^*(t)}{dt} = \frac{m}{N_0+t-1} \tag{9}$$

with initial condition $k_i^*(t_i) = 0$ (note that at $t = t_i$ the newly added node i cannot have incoming edges established through random attachment). The solution to (9) is

$$k_i^*(t) = m\ln\left(\frac{N_0+t-1}{N_0+t_i-1}\right) \tag{10}$$

Moreover, using (3) we also know that for node i with in-degree $k_i(t) = k$

$$\left(\frac{N_0+t-1}{N_0+t_i-1}\right) = \left(\frac{k+\left(1+\frac{1}{p}\right)m}{n+\left(1+\frac{1}{p}\right)m}\right)^{1+\frac{1}{p}} \tag{11}$$

Replacing (11) in (10) we know

$$k_i^* = \left(1+\frac{1}{p}\right)m\ln\left(\frac{k+\left(1+\frac{1}{p}\right)m}{n+\left(1+\frac{1}{p}\right)m}\right)$$

Note that the probability of establishing the third edge that closes the triplet is

$$\left(1+\frac{1}{p}\right) m \ln \left(\frac{k+\left(1+\frac{1}{p}\right) m}{n+\left(1+\frac{1}{p}\right) m}\right) p \qquad (12)$$

For scenario (iii), the number of incoming edges established through triad formation is given by

$$k-\left(1+\frac{1}{p}\right) m \ln \left(\frac{k+\left(1+\frac{1}{p}\right) m}{n+\left(1+\frac{1}{p}\right) m}\right) \qquad (13)$$

which is the probability of establishing the third edge that closes the triplet. Finally, summing (8), (12), and (13) and dividing by the right hand side of (7), we know $C_i(k)$ equals

$$\begin{aligned} & \frac{2\left(k+pm\left(1+\left(1-\frac{1}{p^2}\right)\ln\left(\frac{k+\left(1+\frac{1}{p}\right)m}{n+\left(1+\frac{1}{p}\right)m}\right)\right)\right)}{(k+(1+p)m)\left(k+(1+p)m-1\right)} \\ = \; & \frac{2\left(k+pm+(2+p-\alpha)\ln\left(\frac{k+\varepsilon}{n+\varepsilon}\right)\right)}{(k+p\varepsilon)\left(k+p\varepsilon-1\right)} \end{aligned} \qquad (14)$$

where $\alpha=2+\frac{1}{p}$ and $\varepsilon=(\alpha-1)m$. $\square$

REFERENCES

[1] M. E. J. Newman, "The structure of scientific collaboration networks," *PNAS: Proceedings of the National Academy of Sciences of the United States of America*, vol. 98, no. 2, pp. 404–409, 2001.

[2] J. H. Fowler and S. Jeon, "The authority of Supreme Court precedent," *Social Networks*, vol. 30, no. 1, pp. 16–30, 2008.

[3] C. A. Hidalgo, B. Klinger, A.-L. Barabási, and R. Hausmann, "The product space conditions the development of nations," *Science*, vol. 317, no. 5837, pp. 482–487, 2007.

[4] A. Barrat, M. Barthelemy, R. Pastor-Satorras, and A. Vespignani, "The architecture of complex weighted networks," *PNAS: Proceedings of the National Academy of Sciences of the United States of America*, vol. 101, no. 11, pp. 3747–3752, 2004.

[5] G. Kossinets and D. J. Watts, "Empirical analysis of an evolving social network," *Science*, vol. 311, no. 5757, pp. 88–90, 2006.

[6] J. P. Onnela, A. Chakraborti, K. Kaski, J. Kerész, and A. Kanto, "Dynamics of market correlations: Taxonomy and portfolio analysis," *Physical Review E*, vol. 68, no. 5, p. 056110(12), 2003.

[7] G. H. Hall, A. B. Jaffe, and M. Trajtenberg, "The NBER patent citation data file: Lessons, insights and methodological tools." NBER Working Papers 8498, National Bureau of Economic Research, October 2001.

[8] S. Valverde, R. V. Solé, M. A. Bedau, and N. Packard, "Topology and evolution of technology innovation networks," *Physical Review E*, vol. 76, no. 056118(7), 2007.

[9] P. Moriano and J. Finke, "Heavy-tailed weighted networks from local attachment strategies," in *Proceedings of the IEEE Conference on Decision and Control and the European Control Conference*, Orlando, FL, December 2011.

[10] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Reviews of Modern Physics*, vol. 74, no. 1, pp. 47–97, 2002.

[11] M. E. J. Newman, "The structure and function of complex networks," *SIAM Review*, vol. 45, no. 2, pp. 167–256, 2003.

[12] M. Mitzenmacher, "A brief history of generative models for power law and lognormal distributions," *Internet Mathematics*, vol. 1, no. 2, pp. 226–251, 2004.

[13] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[14] W. J. Reed and B. D. Hughes, "From gene families and genera to incomes and internet file sizes: Why power laws are so common in nature," *Physical Review E*, vol. 66, no. 6, p. 067103(4), 2002.

[15] H. Louch, "Personal network integration: Transitivity and homophily in strong-tie relations," *Social Networks*, vol. 22, no. 1, pp. 45–64, 2000.

[16] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature*, vol. 393, no. 6684, pp. 440–442, 1998.

[17] E. Ravasz and A.-L. Barabási, "Hierarchical organization in complex networks," *Physical Review E*, vol. 67, no. 2, p. 026112(7), 2003.

[18] X. Li and G. Chen, "A local-world evolving network model," *Physica A*, vol. 328, no. 1-2, pp. 274–286, 2003.

[19] P. Holme and B. J. Kim, "Growing scale-free networks with tunable clustering," *Physical Review E*, vol. 65, no. 2, p. 026107(4), 2002.

[20] Z. Zhang, L. Rong, B. Wang, S. Zhou, and J. Guan, "Local-world evolving networks with tunable clustering," *Physica A*, vol. 380, pp. 639–650, 2007.

[21] M. O. Jackson and B. W. Rogers, "Meeting strangers and friends of friends: How random are social networks?," *American Economic Review*, vol. 97, no. 3, pp. 890–915, 2007.

[22] Z. Shao, X. Zou, Z. Tan, and Z. Jin, "Growing networks with mixed attachment mechanisms," *Journal of Physics A: Mathematical and General*, vol. 39, no. 9, pp. 2035–2042, 2006.

[23] P. Moriano and J. Finke, "Power-law weighted networks from local attachments," *Europhysics Letters*, vol. 99, no. 1, p. 18002(6), 2012.

[24] G. Csányi and B. Szendrői, "Structure of a large social network," *Physical Review E*, vol. 69, no. 3, p. 036131(5), 2004.

[25] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. McGraw Hill Higher Education, 4th ed., 2002.